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Graphic Evaluation of Trigonometric Functions Of Complex Variables

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# GRAPHIC EVALUATION OF TRIGONOMETRIC FUNCTIONS OF COMPLEX VARIABLES

BY

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. Truman Lee Kelley

ENTITLED GRAPHIC EVALUATION OF TRIGONOMETRIC FUNCTIONS

OF COMPLEX VARIABLES

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Bachelor of Arts

in Mathematics

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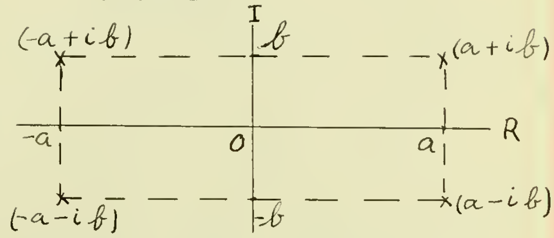
Art. 1.- INTRODUCTION.

The method of the analytic geometry of representing, graphically, numerical values, by distances on a straight line is here assumed. Moreover, we assume the one to one correspondence of the real numbers in a continuum and the points of a straight line.

When numbers of the type $\sqrt{-B}$, called imaginaries, and represented by $\sqrt{-1}\,b$ ($b = \sqrt{B}$), or, simply ib, were introduced, a method was devised for representing them graphically. The common representation of such a number is by measurement of the distance b along a line at right angles to the one used for representing real numbers, and passing through the same origin. ib is represented by a point at distance b, measured upward from th origin and -ib by a point at distance b, below the origin. An extension of the preceding is the representation of a number of the type, (a + ib), by measuring a distance a along the line chosen to represent real numbers, called the axis of reals, and from this point a distance b along a line parallel to the line representing imaginary numbers, called the axis of imaginaries. By this method all real numbers ($\pm$ a), all imaginary numbers ($\pm$ib), and all complex numbers ($\pm$a$\pm$ib), may be represented upon a single plane. The accompanying diagram illustrates this method of representation.

FIGURE I

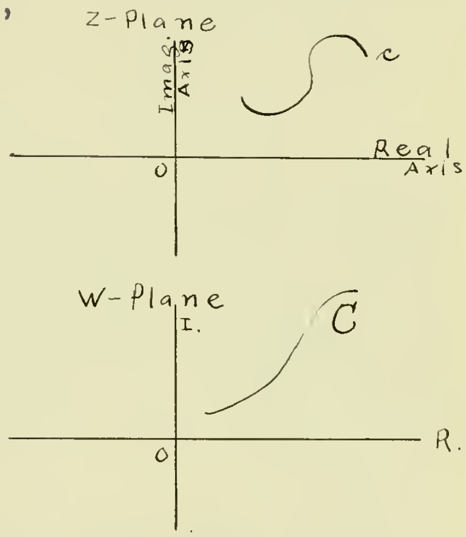



This representation upon the complex plane may be used to represent functions of complex numbers. For convenience two

such planes are used, one to represent the complex variable and the other the function. We shall call the former the Z-plane and the latter the W-plane. The usual method of representing the functional relation between z and W is, $W = f(z)$. It is understood that W and z are, in general, both complex numbers. We may also write, $z = x + iy$, and, $W = U + iV$, in which case U and V are both functions of x,y.

If the independent variable z, in the equation $W = f(z)$, is allowed to take a succession of values, say those represented by points on the curve c, drawn in the Z-plane, then W takes a sequence of values represented, let us assume, by the line C, drawn in the W-plane. By means of the relation, $W = f(z)$, we say that the curve c, of the Z-plane maps into the curve C, of the W-plane.

FIGURE II




If z takes a series of values, represented by a line parallel to the axis of reals, or the axis of imaginaries, this will give some curve in the W-plane. If all the lines parallel to the axis of reals, and those parallel to the axis of imaginaries, in the Z-plane, are thus mapped on the W-plane, the value of W for any z, in $W = f(z)$, may be read by locating the point in the W-plane. The following example will illustrate this:

Given, $W = \frac{1}{z}$. Let us determine what the lines parallel to the x and y axis, respectively, map into in the W-plane:

$$U + iV = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

Equating reals and imaginaries, we have,

$$U = \frac{x}{x^2 + y^2}$$

$$V = \frac{-y}{x^2 + y^2}$$

Eliminating y from these two equations, we have,

$$V^2 + U^2 - \frac{U}{V} = 0, \text{ or}$$

$$V^2 + (U - \frac{1}{2x})^2 = (\frac{1}{2x})^2$$

This equation, for values $x$ = a constant, is represented by a circle passing through the origin, with center upon the U-axis and at a distance $\frac{1}{2x}$ from the origin.

Similarly, we have,

$$U^2 + (V - \frac{1}{2y})^2 = (\frac{1}{2y})^2,$$

which is represented by a circle passing through the origin, with center upon the V-axis and at a distance $\frac{1}{2y}$ from the origin, for $y = c$, a constant. In the accompanying figure, representing the map of $W = \frac{1}{z}$, lines in the Z-plane map into the lines correspondingly lettered in the W-plane.

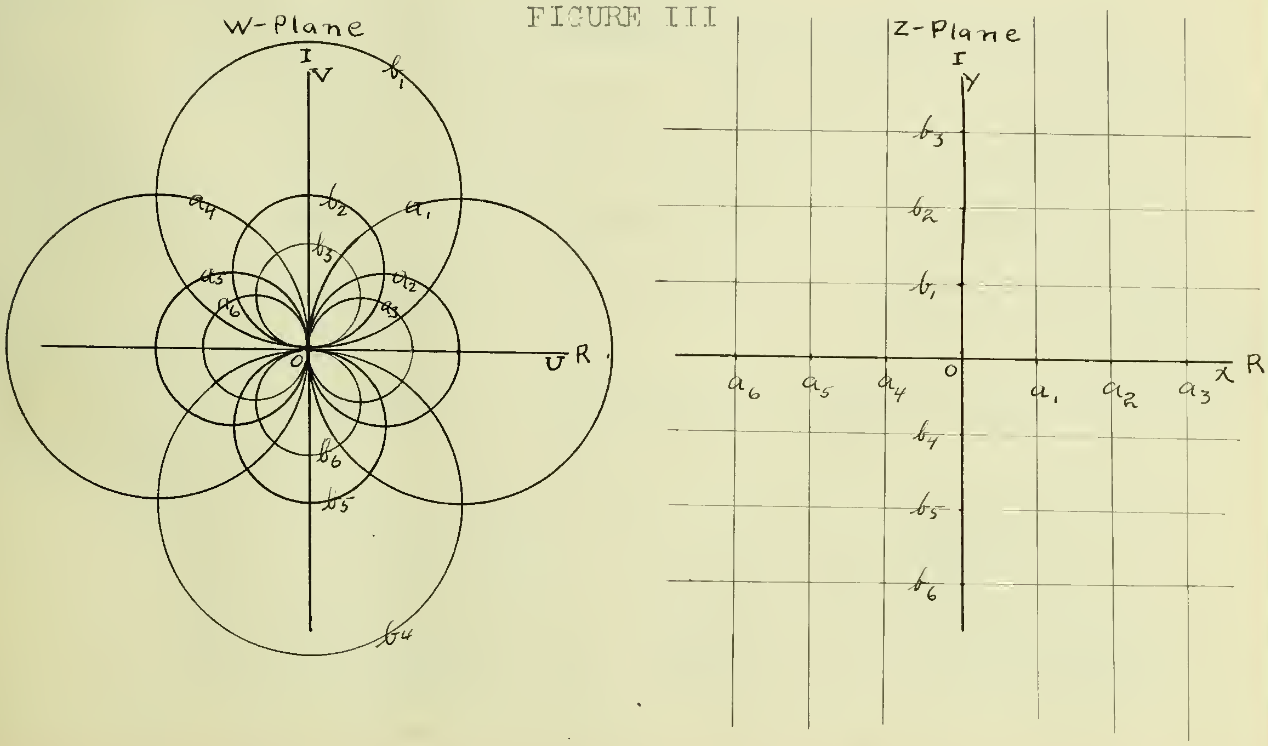


FIGURE III

Art.2.- PROPERTIES OF ANALYTIC FUNCTIONS.

The following propositions, found in any text book dealing with functions of complex variables, are assumed:

(1) The necessary and sufficient condition that W ($W = U + iV$) is an analytic function of z ($z = x + iy$) is that the following partial differential equations are satisfied:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \text{and} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

(2) If W is an analytic function of z, then $U = \psi(x,y)$, and $V = \phi(x,y)$. U and V are conjugate functions of x,y and the two systems of curves are orthogonal.

(3) If $W = f(z)$ is an analytic function, the mapping of the Z-plane into the W-plane is a conformal representation, i.e., infinitesimal elements are preserved.

(4) z is a single-valued and continuous function of W if for every continuous series of points in the Z-plane there is a continuous series of points in the W-plane.

(5) If z is a single-valued and continuous function of W, and W a single valued function of z, then there exist fundamental regions in the z-plane, such that each map once and only once into the entire W-plane. In this case, $W=f(z)$, is a periodic function and the limits of the fundamental region are determined by the period of f(z).

(6) The trigonometric functions of complex numbers are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cot z = \frac{1}{\tan z}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sec z = \frac{1}{\cos z}$$

$$\tan z = \frac{\sin z}{\cos z} \qquad \csc z = \frac{1}{\sin z}$$

Art. 3.- DISCUSSION OF W = SIN Z.

Applying the tests stated in Art. 2, to W = sin z, we have:

(1) $$U + iV = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2i}$$

$$= \frac{e^{y} + e^{-y}}{2} \sin x + i \frac{e^{y} - e^{-y}}{2} \cos x$$

Equating reals and imaginaries, we have,

$$U = \frac{e^{y} + e^{-y}}{2} \sin x; \quad V = \frac{e^{y} - e^{-y}}{2} \cos x, \text{ and}$$

$$\frac{\partial U}{\partial x} = \frac{e^{y} + e^{-y}}{2} \cos x; \quad \frac{\partial V}{\partial y} = \frac{e^{y} + e^{-y}}{2} \cos x$$

$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$, and the function is analytic.

Since $\frac{e^{y} + e^{-y}}{2}$ = hyberbolic cosine y, and

$\frac{e^{y} - e^{-y}}{2}$ = hyberbolic sine y, we can write,

$$U = \cosh y \sin x$$

$$V = \sinh y \cos x$$

(2) The two systems of curves,

$$U = \cosh y \sin x$$

$$V = \sinh y \cos x$$

are orthogonal, or, stated otherwise, U and V are conjugate functions of x and y.

(3) The mapping of the Z-plane into the W-plane is a conformal representation.

(4) Since cosh y is a single-valued and continuous function for all values of y within the limits, $0 \leqq y < \infty$, and sin x is a single-valued and continuous function for all values of x in the interval, $\frac{(2n+1)\pi}{2} < x \leqq \frac{(2n+3)\pi}{2}$. (n is an integer).

Therefore,

$$U = \cosh y \sin x,$$

is a single-valued and continuous function of U within the region,

$$0 \leqq y < \infty \ , \qquad \frac{(2n+1)\pi}{2} \leqq x \leqq \frac{(2n+3)\pi}{2} .$$

Similarly, since sinh y is a single-valued and continuous function for all values of y and cos x is a single-valued and continuous function for all values of x within the interval, $n\pi \leqq x \leqq (n+1)\pi$, therefore,

$$V = \sinh y \cos x,$$

is a single valued and continuous function of V within the region

$$-\infty < y < \infty \ , \qquad n\pi \leqq x \leqq (n+1)\pi.$$

Following directly from the above,

$$W = U + iV,$$

is a single-valued and continuous function of x,y within the common region. That is, W is a single-valued and continuous function of both x and y within the region,

$$0 \leqq y < \infty \ , \qquad n\frac{\pi}{2} \leqq x \leqq (n+1)\frac{\pi}{2} .$$

(5) To determine the period of sin z, let

$$\sin z = \sin (z + \alpha), \text{ where } \alpha = a + ib.$$

Solving and equating reals and imaginaries, as follows:

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(z+\alpha)} - e^{-i(z+\alpha)}}{2i}$$

$$\frac{e^{ix-y} - e^{-ix+y}}{2i} = \frac{e^{i(x+a)-(y+b)} - e^{-i(x+a)+(y+b)}}{2i}$$

$$\frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2i}$$

$$= \frac{e^{-(y+b)}\left\{\cos(x+a) + i \sin(x+a)\right\} - e^{y+b}\left\{\cos(x+a) - i \sin(x+a)\right\}}{2i}$$

$$\frac{(e^{-y} - e^{y})}{2i} \cos x + \frac{(e^{-y} + e^{y})}{2} \sin x$$

$$= \frac{e^{-(y+b)} - e^{y+b}}{2i} \cos(x+a) + \frac{e^{-(y+b)} + e^{y+b}}{2} \sin(x+a)$$

$$\sinh y \cos x = \sinh (y+b) \cos(x+a)$$

$$\cosh y \sin x = \cosh (y+b) \sin(x+a)$$

$$\frac{\sinh y}{\sinh(b+y)} = \frac{\cos(x+a)}{\cos x}$$

$$\frac{\cosh y}{\cosh(b+y)} = \frac{\sin(x+a)}{\sin x}$$

Since the hyperbolic functions are not periodic, these equations can only hold when $b = 0$, and $\sin(x+a) = \sin x$ & $\cos(x+a) = \cos x$. That is to say, $a = 2\pi k$ ( $k$ an integer), and the period of $\sin z$ equals th period of $\sin x$.

Art.4.- MAPPING OF SIN Z.

(1) Fundamental region.

Let us choose a region in the Z-plane with the following boundary:

$$-\infty < y < \infty , \qquad \frac{-\pi}{2} \leqq x \leqq \frac{\pi}{2} ,$$

and map this boundary of z and a few intermediate points, by means of the equations for U and V derived in Art.1.

$$U = \cosh y \sin x \qquad (1)$$

$$V = \sinh y \cos x \qquad (2)$$

Consider a point in the Z-plane (see Fig.IV) such as a, where x is a constant, say equal to $-\frac{\pi}{2}+\epsilon$ , $\epsilon$ being an arbitrarily small number and y is negative, say equal to $-k$. Substituting these values in equation (1), we have,

$$U = \cosh(-k) \sin(-\tfrac{\pi}{2}+\epsilon)$$

$= (+K)(-1+\eta) < -1$ ($\cosh(-k) = K$, a positive number considerably $> 1$ if $|k|$ is appreciable $> 0$, and $\sin(-\frac{\pi}{2}+\epsilon) = (-1+\eta)$, $\eta$ being a small number $< \epsilon$ .)

or, simply,

$$U < -1.$$

Substituting in equation (2), we have,

$V = \sinh(-k)\cos\left(-\frac{\pi}{2}+\epsilon\right)$

$= (-K')(\eta) = -\delta$, $\quad (\sinh(-k) = -K'$, and $\cos\left(-\frac{\pi}{2}+\epsilon\right) = \eta$,

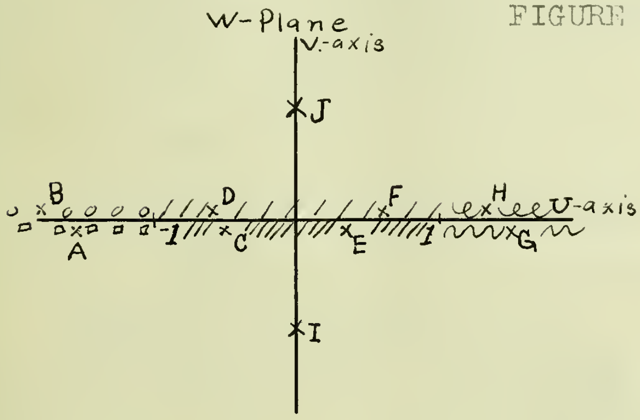
or, simply,

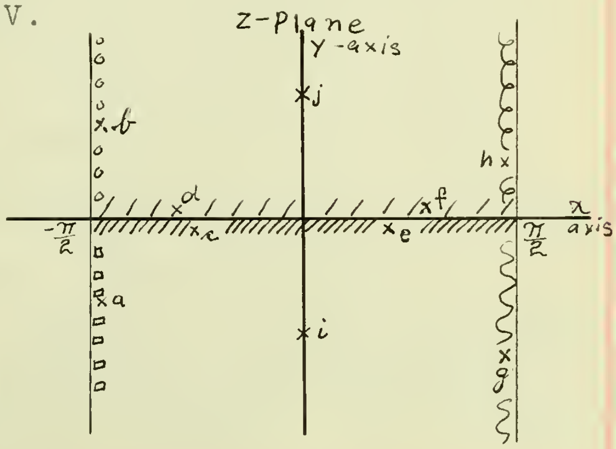
$V = -\delta$

These values, $U < -1$, and $V = -\delta$, are represented in the W-plane by a point such as A (see Fig.IV). Similarly points b, c, etc. map into points B, C, etc., as shown in the accompanying table and figure.

| Let x= | Let y= | Then U= | and V= | Represented on drawings by points such as (Z-Plane) | W-Plane |
|---|---|---|---|---|---|
| $-\frac{\pi}{2}+\epsilon$ | $-$ | $<-1$ | $-\delta$ | a | A |
| " | $+$ | $<-1$ | $+\delta$ | b | B |
| $<0$ & $>-\frac{\pi}{2}$ | $-\epsilon$ | $<0$ & $>-1$ | $-\delta$ | c | C |
| " | $+\epsilon$ | " | $+\delta$ | d | D |
| $>0$ & $<\frac{\pi}{2}$ | $-\epsilon$ | $>0$ & $<1$ | $-\delta$ | e | E |
| " | $+\epsilon$ | " | $+\delta$ | f | F |
| $\frac{\pi}{2}-\epsilon$ | $-$ | $>1$ | $-\delta$ | g | G |
| " | $+$ | " | $+\delta$ | h | H |
| $0$ | $-$ | $0$ | $-$ | i | I |
| $0$ | $+$ | $0$ | $+$ | j | J |

FIGURE IV.



This mapping shows that the region in the Z-plane, bounded as follows;

$$\# \quad -\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}\,; \qquad 0 \leqq y$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}\,; \qquad 0 > y,$$

is a fundamental region, that is, maps into the entire W-plane.

(2) Relation between the four quadrants.

Let us consider some point, say, a ib, in the first quadrant of the indicated fundamental region of the Z-plane. We have,

| | | | |
|---|---|---|---|
| $z = a + ib$ | $U = \cosh b \sin a = A$ | $V = \sinh b \cos a = B$ | $W = A + iB$ |

Similarly for the other quadrants,

| | | | |
|---|---|---|---|
| $z = -a + ib$ | $U = \cosh b \sin(-a) = -A$ | $V = \sinh b \cos(-a) = B$ | $W = -A + iB$ |
| $z = -a - ib$ | $U = \cosh -b) \sin(-a) = -A$ | $V = \sinh -b) \cos(-a) = -B$ | $W = -A - iB$ |
| $z = a - ib$ | $U = \cosh(-b) \sin a = A$ | $V = \sinh(-b) \cos a = -B$ | $W = A - iB$ |

That is, the sine of any complex number, $-a + ib$, in the second quadrant, is equal to the sine of $a + ib$ in the first quadrant, with the sign before the real element changed to minus. The sine of any complex number, $-a - ib$, in the third quadrant, is equal to the sign of $a + ib$ with the signs of both the imaginary and the real elements changed to minus. The sine of any complex number, $a - ib$, in the fourth quadrant, is equal to the sign of $a + ib$ with the sign of the imaginary element changed to minus.

These simple relations give us a method for determining the sines of complex numbers in the second, third, and fourth quadrants if those of the first are known. It will therefore be

---

\# These two regions together comprise the fundamental region. The equality signs are left off of the limits for x in the one case because the entire line, $x = \frac{\pi}{2}$, would map twice into the U-axis from 1 to $\infty$. Likewise, but half of the line $x = -\frac{\pi}{2}$, is used, in order that it shall map but once into the U-axis from -1 to $-\infty$.

sufficient to map the first quadrant only.

If the lines $x = c$ ($c \equiv$ a constant), and $y = c$, in the Z-plane, are mapped into the W-plane, the value of the sine of any complex number, $a + ib$, may be found by reading the values of U and V in the W-plane, where the mappings of $x = a$ and $y = b$ intersect.

(3) <u>The mapping of lines parallel to the axes.</u>

To map the lines, $x = c$, we eliminate y from equations,

$U = \cosh y \sin x$

$V = \sinh y \cos x$,

and to map the lines, $y = c$, we eliminate x from the same equations, as follows:

$U = \cosh y \sin x$ , $\qquad V = \sinh y \cos x$

$\frac{U^2}{\cosh^2 y} = \sin^2 x$ , $\qquad \frac{V^2}{\sinh^2 y} = \cos^2 x$

Adding, we have,

$$\frac{U^2}{\cosh^2 y} + \frac{V^2}{\sinh^2 y} = 1 \qquad (1)$$

Again, we get,

$\frac{U^2}{\sin^2 x} = \cosh^2 y$, $\qquad \frac{V^2}{\cos^2 x} = \sinh^2 y$

Subtracting, we have,

$$\frac{U^2}{\sin^2 x} - \frac{V^2}{\cos^2 x} = 1 \qquad (2)$$

Plotted upon rectangular coordinates the following equation;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

represents an ellipse, intercepting the axes at $x = \pm a$, and $y = \pm b$, and with foci at $\pm\sqrt{a^2-b^2}$. Equation (1) is of this type, in

which cosh y = a, sinh y = b, U = x, and V = y. Therefore, if y is a parameter equation (1) represents a system of ellipses, intercepting the U and V axes at $\pm$ cosh y, and $\pm$ sinh y respectively, and having foci at $\pm$ 1, since $\pm\sqrt{\cosh^2 y - \sinh^2 y} = \pm 1$.

Again, the equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

represents an hyperbola, intercepting the x-axis at $x = \pm a$, and having foci at $\pm\sqrt{a^2 + b^2}$. Equation (2) is of this type, in which sin x = a, cos x = b, U = x, and V = y. Therefore, if x is a parameter, equation (2) represents a system of hyperbolas, intercepting the U-axis at $\pm$ sin x and having foci at $\pm$ 1, since $\pm\sqrt{\sin^2 x + \cos^2 x} = \pm 1$.

In equation (1), as y takes in turn, larger and larger constant values, cosh y and sinh y take ever increasing values and the intersections of the ellipses with the axes become more and more distant from the origin.

In equation (2), as x takes in turn larger and larger constant values, always $< \frac{\pi}{2}$, sin x increases up to 1 as a limit, that is, the intersections of the hyperbolas with the U axis travel from 0 to 1.

The nature of this mapping for the first quadrant is shown in map I.

Art. 5.- EXPLANATION OF THE MAPS.

(1) General description of all the maps.

The cross-ruled paper upon which the mapping is made represents the first quadrant of the W-plane. As explained in Art. 4, the mapping for one quadrant is sufficient to determine

the values for the entire plane. For values of U, distances along the horizontal axis are measured, and for values of V, distances along the vertical axis. The numbering in green ink indicates the values of U and V. The ellipses are the mappings of the lines in the Z-plane, $y =$ a constant. The value of this constant is indicated by the numbering in black ink along the left and part of the lower margin. The first map is constructed for values of y as follows:

$$0 \leqq y \leqq 2.30$$

The second for,

$$2.30 \leqq y \leqq 3.90$$

The third for,

$$3.90 \leqq y \leqq 5.50$$

Art.6 gives a method for finding the values of the sines of complex numbers in which the imaginary element is $> 5.50$.

The hyperbolas are the mappings of the lines in the Z-plane, $x =$ a constant. The value of this constant is indicated by the numbering in black ink near the origin and along the boundary most distant from the origin. In all the maps,

$$0 \leqq x \leqq \frac{\pi}{2}.$$

(2) Map I.- ( $0 \leqq y \leqq 2.30$ )

The scale in this map is, 1 mm. = .02, for values of U and V. It is to be noticed that the mapping grows dense as we approach the point, $U = 1$, $V = 0$, and for this reason fewer ellipses and hyperbolas are plotted in the region near this point than in the more distant regions. This is true of all the maps. The following outline gives the difference in the values of y for adjacent ellipses that are mapped, and the difference in the

value of x for adjacent hyperbolas that are mapped in the various regions of map I. Similar outlines will be found for maps II and III.

| | |
|---|---|
| For region bounded as follows: | : |
| $\frac{\pi}{2} > x > 1$<br>$.5 > y > 0$ | )Adjacent ellipses differ by .1.<br>)Adjacent hyperbolas differ by .05. |
| $1 > x > .5$<br>$.5 > y > 0$<br>$\frac{\pi}{2} > x > 1$<br>$1 > y > .5$ | )Adjacent ellipses differ by .05<br>) and<br>)Adjacent hyperbolas differ by .05 |
| $.5 > x > 0$<br>$.5 > y > 0$ | )Adjacent ellipses differ by .05<br>)Adjacent hyperbolas differ by .02 |
| In the remaining region | )Adjacent ellipses differ by .02<br>)Adjacent hyperbolas differ by .02 |

(3) Map II. $(2.30 \leqq y \leqq 3.90)$

The scale in this map for values of U and V is 1 mm.= .1.

| | |
|---|---|
| For region bounded as follows: | : |
| $2.30 < y < 3.00$ | )Adjacent ellipses differ by .05<br>)Adjacent hyperbolas differ by .05 |
| $3.00 < y < 3.90$ | )Adjacent ellipses differ by .02<br>)Adjacent hyperbolas differ by .02 |

Note that the ellipses are practically circles and the hyperbolas are nearly straight lines in this map.

(4) Map III. $(3.90 \leqq y \leqq 5.50)$

The scale in this map for values of U and V is 1 mm. .5.

| | |
|---|---|
| For region bounded as follows: | : |
| $3.90 < y < 4.50$ | )Adjacent ellipses differ by .05<br>)Adjacent hyperbolas differ by .02 |
| $4.50 < y < 5.50$ | )Adjacent ellipses differ by .02<br>)Adjacent hyperbolas differ by .02 |

In this map the ellipses, as constructed, are no longer

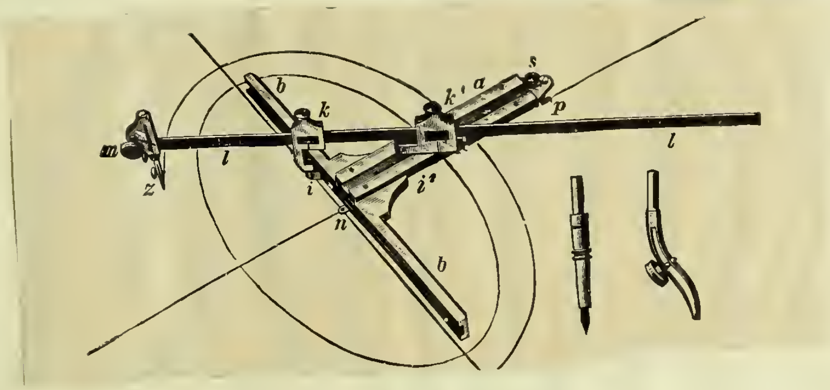
ellipses, but circles, having their center at the origin, and the hyperbolas, as drawn are straight lines, intersecting, if produced, at the origin. (See Art. 6 (a))

(5) Description of ellipsograph.

The ellipses in the maps were drawn with the help of an ellipsograph. The ellipsograph used consists of two main parts, the frame and the beam compass. The former forms a "T", the arms and stem of which are to lie along the major and minor axes, respectively. In both arms and stem are slots, in which clamps, bearing the beam compass, are free to slide. To set the instrument to draw a given ellipse the clamp running in the slot along the arms is clamped to the beam of the beam compass so that the distance between the pen point and the clamp is equal to one-half the minor axis of the desired ellipse. Similarly, the distance between the other clamp and the pen point is made equal to one-half the major axis. Under these conditions the pen point will trace the desired ellipse, because of the fact that for a given ellipse, if any line cuts the circumference at right angles the distance intercepted by the axes is constant.

Accompanying is a cut showing the instrument, taken from the catalog of Eugene Dietzgen Company.

FIGURE V



(6) Construction of the maps.

In the construction of the ellipses all the data necessary was the values of the major and minor axes. From the equation,

$$\frac{U^2}{\cosh^2 y} + \frac{V^2}{\sinh^2 y} = 1$$

we see that the intersection of the ellipse with the U-axis (or major axis) is at the point $U = \cosh y$, and that the intersection with the V-axis (or minor axis) is at the point $V = \sinh y$. Cosh y and sinh y were obtained from a table of hyperbolic functions#. The ellipses for the three maps were drawn in this manner, but it is to be noticed that for the greater part of the second map and for all of the third the difference between cosh y and sinh y is less than .05, or, expressed in millimeters, less than .5 mm. in map II and less than .1 in map III. In the case of map III no attempt was made to express this slight difference.

The hyperbolas were constructed by drawing a smooth curve through a certain number of calculated points. For map I the intersections of the various hyperbolas with the ellipses $y = 0$ (this is simply the U axis from 0 to 1), $y = .50$, $y = 1.00$, $y = 1.50$, and $y = 2.25$ were calculated from the following equations:

$$U = \cosh y \sin x$$

$$V = \sinh y \cos x$$

For map II the intersections of the hyperbolas with the ellipses, $y = 2.30$, $y = 3.00$ and $y = 3.90$ were sufficient. For map

# For tables consulted see Bibliography.

III the intersections of the hyperbolas with the single ellipse $y = 5.50$ were all that were needed, because the system of hyperbolas is orthogonal to the system of ellipses and the ellipses in this map are not distinguishable from circles, therefore the orthogonal rays, if produced, would pass through the origin, so the origin is available as a second point in determining the hyperbolas. (See Art.6 (a) ).

Art.6.- EVALUATION OF SIN $a+ib$, WHERE $b > 5.5$.

The accuracy with which values may be read in any of the maps is less than $\frac{1}{4}$ mm. This corresponds to a value of .125 in U and V in map III and to a value of .025 in U and V in map II. The values of U and V are;

$$U = \cosh Y \sin x$$

$$V = \sinh y \cos x,$$

$$\text{or,} \quad U = \frac{e^y + e^{-y}}{2} \sin x$$

$$V = \frac{e^y - e^{-y}}{2} \cos x$$

$$U = \frac{e^y}{2} \sin x + \frac{e^{-y}}{2} \sin x$$

$$V = \frac{e^y}{2} \cos x - \frac{e^{-y}}{2} \cos x$$

If y is so large that,

$$\frac{e^{-y}}{2} \sin x < .025$$

$$\frac{e^{-y}}{2} \cos x < .025,$$

then U and V may be written,

$$U = \frac{e^y}{2} \sin x$$

$$V = \frac{e^y}{2} \cos x,$$

without introducing an error which affects the graph to a noticeable degree. When $y = 3$.

$$\frac{e^{-y}}{2} \sin \frac{\pi}{2} = .024895$$

$$\frac{e^{-y}}{2} \cos 0 = .024895$$

Therefore the following relations hold for our purposes:

(a)
$$\frac{U}{\left(\frac{e^y}{2}\right)^2} + \frac{V}{\left(\frac{e^y}{2}\right)^2} = 1$$

$$U^2 + V^2 = \frac{e^y}{2}\ 2$$

This is the form which the equation of the ellipses takes when $\frac{e^{-y}}{2}$ is so small that we can neglect it. It is the equation of a circle with radius $\frac{e^y}{2}$ and center at the origin. The orthogonal system has this equation;

$$\frac{U^2}{\sin^2 x} - \frac{V^2}{\cos^2 x} = \frac{e^y}{2} - \frac{e^y}{2} = 0$$

$$U = (\tan x)\ V,$$

which is the equation of a system of straight lines intersecting at the origin.

(b) Given, $U + iV = \sin (x + iy)$. If $y > 3$

$$U = \frac{e^y}{2} \sin x$$

$$V = \frac{e^y}{2} \cos x$$

$$10\ (U + iV) = \sin (x' + iy')$$

$$10U = \frac{e^{y'}}{2} \sin x'$$

$$10V = \frac{e^{y'}}{2} \cos x'$$

$$U = \frac{e^{y'}}{20} \sin x' = \frac{e^y}{2} \sin x$$

$$V = \frac{e^{y'}}{20} \cos x' = \frac{e^y}{2} \cos x$$

Dividing, we have,

$$\frac{\sin x'}{\cos x'} = \frac{\sin x}{\cos x}$$

$$\tan x' = \tan x$$

$$x' = x$$

Therefore,

$$\frac{e^{y'}}{20} = \frac{e^y}{2}$$

$$\frac{e^{y'}}{10} = e^y$$

$$e^{y'} = 10e^y = e^{y+2.3026-}$$

$$y' = y + 2.3026-$$

and,, $10\ (U + iV) = \sin\left[x + i(y + 2.3026-)\right] = 10 \sin\ (x + iy)$ (B)

That is, the sines of numbers, $a + ib$, where $b > 5.5$, the largest value for which the maps are constructed, may be obtained by finding the value of $\sin\left[a + i(b - 2.3026)\right]$, and multiplying by 10. If $(b - 2.3026) \not< 5.5$, look up the sine of $\left[a + i(b - n2.3026)\right]$ and multiply by $10^n$. n is a positive integer and must be chosen so that $3. \leqq (b - n\ 2.3026) \leqq 5.5$.

Art.7.- Evaluation of $\sin a + ib$, where $-\frac{\pi}{2} > a$, or $> \frac{\pi}{2}$.

From Art.3 (5) we have the following relation:

$$\sin\ (a + ib) = \sin\ (2\pi k + a + ib)$$

$$U = \cosh b \sin a = \cosh b \sin(a + 2k\pi) = \cosh b(-1)^k \sin\ (a + k\pi)$$

$$V = \sinh b \cos a = \sinh b \cos(a + 2k\pi) = \sinh b(-1)^k \cos\ (a + k\pi),$$

where k is a positive or negative integer, so determined that $-\frac{\pi}{2} \leqq (a + k\pi) \leqq \frac{\pi}{2}$. Therefore, considering absolute values only, of U and V,

$$|U| + i\,|V| = \sin(a + ib)$$

$$|U| + i\,|V| = \sin(a + k\pi + ib)$$

The sign of U is the same as the sign of $(-1)^k \sin(a + k\pi)$, and

the sign of V is $(-1)^k$, when b is positive, in the equation $U + iV = \sin a + ib$. Or, in general, the sign of U is the same as the sign of $(-1)^k \sin(a + k\pi)$, and the sign of V is the same as the sign of $(-1)^k b$. By this means we can relate the sine of any complex number with some other complex number lying within $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for the real part.

In Art. 4 (2) the relation between the four quadrants is given, so we can now express the sine of any complex number in terms of the sine of some complex number in the first quadrant. This relation is expressed by the following equations, in which a and b are any real numbers, and k an integer so determined that $-\frac{\pi}{2} < (a + k\pi) < \frac{\pi}{2}$.

$$A \quad \left( \begin{array}{l} \sin(a + ib) = U + iV \\ \sin(|a + k\pi| + i|b|) = u + iv \\ U = (-1)^k \dfrac{(a + k\pi)}{(|a + k\pi|)} u \\ V = (-1)^k \dfrac{b}{|b|} v \end{array} \right.$$

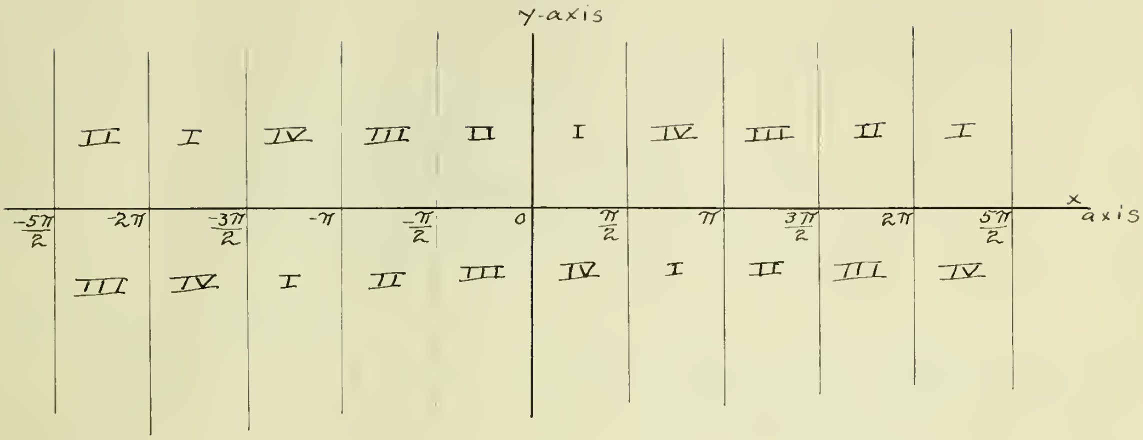
This relation, together with,

$$B \quad ( \sin(a + ib) = 10 \sin\left[a + i(b - n\ 2.302585)\right] ,$$

needed only when $b > 5.5$, enables us to find the sine of any complex number to the degree of accuracy for which the maps are constructed.

The following drawing shows the regions in the Z-plane mapping into the same regions in the W-plane as the four quadrants in our chosen fundamental region:

FIGURE VI




Art.8.- ACCURACY OF THE MAPS.

The precision with which the maps may be read accurately is practically to the nearest $\frac{1}{2}$ mm. The errors of construction probably do not exceed .3 mm. and with careful reading an error greater than .3 mm. should not be made on this account. Therefore in map I the values of U and V are correct to $\pm$ .01, in map II to $\pm$ .05, and in map III to $\pm$ .3. The percentage error in the value of the modulus varies in all the maps inversely as the distance from the origin, being, approximately, one percent at a distance of 5 cm. and .2 percent at a distance of 25 cm. In the following article the actual error and the percentage error is shown for several examples.

The percentage error in the modulus is calculated by finding the modulus, obtained from the values of U and V as given by the maps, and comparing with the true modulus, calculated from the values of U and V from the equations,

$$U = \cosh y \sin x$$

$$V = \sinh y \cos x$$

The calculation of the percentage error in a single case, having

given tables of the hyperbolic functions and the ordinary trigonometric functions, is somewhat of a task, involving 12 operations, as follows:

( 2 multiplications
( 4 squarings
( 2 additions
( 2 extractions of square root
( 1 subtraction
( 1 division

Art. 9.- EXAMPLES.

Let it be desired to find the sine of (1.320 + i 2.024). Turn to map I. Follow along either the U or V-axis until 2.00 is reached. The next ellipse is 2.02 and the next 2.04. Estimate $\frac{4}{20}$ of the distance between 2.02 and 2.04 and follow the ellipse to the point where it intersects with the hyperbola 1.32, found most readily by referring to the right boundary of the map. Keeping this point of intersection follow down the green lines to the lower boundary and determine the value of U by reference to the numbering in green ink. This value is found to be approximately 3.731. From the same point of intersection follow the green lines to the left or right boundary and determine the value of V by reference to the numbering in green ink along these boundaries. This value is found to be approximately .922. This gives,

$$\sin(1.320 + i\ 2.024) = 3.731 + i\ .922$$

The correct value is,

$$\sin(1.320 + i\ 2.024) = 3.730 + i\ .923,$$

showing an error in the modulus of .0008, or .02 per cent.

The examples in the following tables are chosen at random:

| | Sine of the number | Values from maps U | V | Correct values U | V | % error in moduli |
|---|---|---|---|---|---|---|
| | .721 + i .362 | .702 | .270 | .70386 | .2779 | .6 |
| Map I | .445 + i2.241 | 2.050 | 4.201 | 2.0469 | 4.2155 | .5 |
| $y < 2.3$ | .750 + i .750 | .894 | .594 | .8825 | .6017 | .6 |
| | 1.320 + i2.024 | 3.731 | .922 | 3.7299 | .9230 | .02 |
| Map II | .252 + i2.860 | 2.17 | 8.41 | 2.184 | .8427 | .2 |
| $2.3 < y < 3.9$ | .755 + i3.427 | 10.57 | 11.28 | 10.559 | 11.198 | .4 |
| | 1.444 + i3.860 | 23.56 | 3.02 | 23.553 | 3.001 | .04 |
| Map III | .790 + i4.463 | 30.75 | 30.60 | 30.817 | 30.522 | .02 |
| $3.9 < y < 5.5$ | 1.245 + i4.988 | 69.65 | 23.54 | 69.468 | 23.469 | .5 |
| | .362 + i5.456 | 41.32 | 108.90 | 41.466 | 109.503 | .6 |

| See footnote | Principle used | To obtain sine of number below | Look up sine of this number | Values from maps u | v |
|---|---|---|---|---|---|
| 1 | B | 1.06 + i6.104 | 1.06 + i3.801 | 19.56 | 10.99 |
| | | sin(1.06 + i6.104) = 195.6 + i109.9 | | | |
| 2 | B | .53 + i7.593 | .53 + i5.290 | 50.25 | 85.72 |
| | | sin(.53 + i7.593) = 502.5 + i857.2 | | | |
| 3 | A | 2.242 + i2.024 | .900 + i2.024 | 3.020 | 2.315 |
| | | sin(2.242 + i2.024) = 3.020 - i2.315 | | | |
| 4 | A | 3.500 + i4.00 | .358 + i4.00 | 9.50 | 25.50 |
| | | sin(3.500 + i4.00) = -9.50 - i2.315 | | | |
| 5 | A and B | 5.802 + i28.000 | .481 + i4.974 | 33.50 | 64.00 |
| | | sin(5.802 + i28.) = $10^{10}$(-33.50 + i64.) | | | |
| 6 | A and B | 7. + i10. | .717 + i5.395 | 72.42 | 83.00 |
| | | sin (7 + i10) = 7242. + i8300. | | | |

(Table Continued)

| See footnote | Principle used | To obtain sine of number below | Look up sine of this number | Values from maps u | v |
|---|---|---|---|---|---|
| 7 | A | $-4.+i4.42$ | $.858+i4.42$ | 31.49 | 27.37 |
| | | $\sin(-4.+i4.42) = 31.49 - i27.37$ | | | |
| 8 | A and B | $-17.908 - i34.446$ | $.942+i4.512$ | 36.67 | 26.85 |
| | | $\sin(-17.908 - i34.446) = 10^{13}(36.67 - i26.85)$ | | | |

1 $\sin(1.06+i6.104) = 10^1\sin\left[1.06+i(6.104-(1)2.303)\right]$

$= 10\ \sin(1.06+i3.801)$

2 $\sin(.53+i7.593) = 10^1\sin\left[.53+i(7.593-(1)2.303)\right]$

$= 10\ \sin(.53+i5.290)$

3 $\sin(2.242+i2.024 = U+iV$

$\sin(|2.241+(-1)\pi| + i2.024) = u+iv$

$U = (-1)^{-1}\ \frac{(2.242-\pi)}{|2.242-\pi|}\ u = u$

$V = (-1)^{-1}\ \frac{(2.024)}{|2.024|}\ v = -v$

4 $\sin(3.5+i4.) = U+iV$

$\sin(|3.5+(-1)\pi| + i4.) = u+iv$

$U = (-1)^{-1}\ \frac{(3.5-\pi)}{|3.5-\pi|}\ u = -u$

$V = (-1)^{-1}\ \frac{4}{|4|}\ v = -v$

5 $\sin(5.802+i28.) = U+iV$

$\sin(|5.802+(-2)\pi| + i\left[28.-(10)2.3026\right])$

$= \sin(|-.481| + i4.974) = u+iv$

$U = (10)^{10}(-1)^2\ \frac{-.481}{|-.481|}\ u = -10{,}000{,}000{,}000u$

$V = (10)^{10}(-1)^2\ \frac{4.974}{|4.974|}\ v = 10{,}000{,}000{,}000v$

6 $\sin(7. + i10.) = U + iV$

$\sin(|7. + (-2)\pi| + i[10 - (2)2.3026]\ )$

$= \sin(.717 + i5.395) = u + iv$

$U = (10)^2(-1)^{-2}\ \frac{.717}{|.717|}\ u = 100u$

$V = (10)^2(-1)^{-2}\ \frac{10}{|10|}\ v = 100v$

---

7 $\sin(-4. + i4.42) = U + iV$

$\sin(|-4. + (1)\pi| + i4.42)$

$= \sin(|-.858| + i4.42) = u + iv$

$U = (-1)^1\ \frac{-.858}{|-.858|}\ u = u$

$V = (-1)^1\ \frac{4.42}{|4.42|}\ v = -v$

---

8 $\sin(-17.908 - i34.446) = U + iV$

$\sin(|-17.908 + (6)\pi| + i|-[34.446 - (13)2.30259]\ )$

$= \sin(|.942| + i|-4.512|) = u + iv$

$U = (10)^{13}(-1)^6\ \frac{.942}{|.942|}\ u = 10{,}000{,}000{,}000{,}000u$

$V = (10^{13}(-1)^6\ \frac{-34.446}{|-34.446|}\ v = -10{,}000{,}000{,}000{,}000v$

---

For convenience of reference the following tables, giving multiples of $\pi$ and of 2.3035850 up to 20, are subjoined:

| Multiplier | $\times \pi$ |
|---|---|
| 1 | 3.14159265 |
| 2 | 6.283 |
| 3 | 9.425 |
| 4 | 12.566 |
| 5 | 15.708 |
| 6 | 18.850 |
| 7 | 21.991 |
| 8 | 25.133 |
| 9 | 28.274 |
| 10 | 31.416 |
| 11 | 34.558 |
| 12 | 37.699 |
| 13 | 40.841 |
| 14 | 43.982 |
| 15 | 47.124 |
| 16 | 50.265 |
| 17 | 53.407 |
| 18 | 56.549 |
| 19 | 59.690 |
| 20 | 62.832 |

| Multiplier | $\times \log_e 10$ |
|---|---|
| 1 | 2.30258509 |
| 2 | 4.605 |
| 3 | 6.908 |
| 4 | 9.210 |
| 5 | 11.513 |
| 6 | 13.816 |
| 7 | 16.118 |
| 8 | 18.421 |
| 9 | 20.723 |
| 10 | 23.026 |
| 11 | 25.328 |
| 12 | 27.631 |
| 13 | 29.934 |
| 14 | 32.236 |
| 15 | 34.539 |
| 16 | 36.841 |
| 17 | 39,144 |
| 18 | 41.447 |
| 19 | 43.749 |
| 20 | 46.052 |

Art.10.- EVALUATION OF COS z, TAN z, COT z, SEC z, AND CSC z.

(1) Cos z.

$$W = U + iV = \cos z = \cos(x+iy) = \frac{e^{ix-y} + e^{-ix+y}}{2}$$

$$= \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x + i \sin x)}{2}$$

$$= \frac{e^{-y} - e^{y}}{2} \cos x + i \frac{e^{-y} + e^{y}}{2} \sin x$$

$$= \sinh y \cos x + i \cosh y \sin x$$

$$U = \sinh y \cos x, \qquad V = \cosh y \sin x$$

Observing that in, $\sin(x+iy)$,

$V = \sinh y \cos x$, $\qquad U = \cosh y \sin x$, we see that all we have to do to find the cosine of a complex number is to look up the sine and interchange the real and imaginary elements.

(2) Tan z, cot z, sec z, csc z.

Having sin z and cos z the ordinary trigonometric relations give us the other functions:

$$\tan z = \frac{\sin z}{\cos z}$$

$$\cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}$$

$$\csc z = \frac{1}{\sin z}$$

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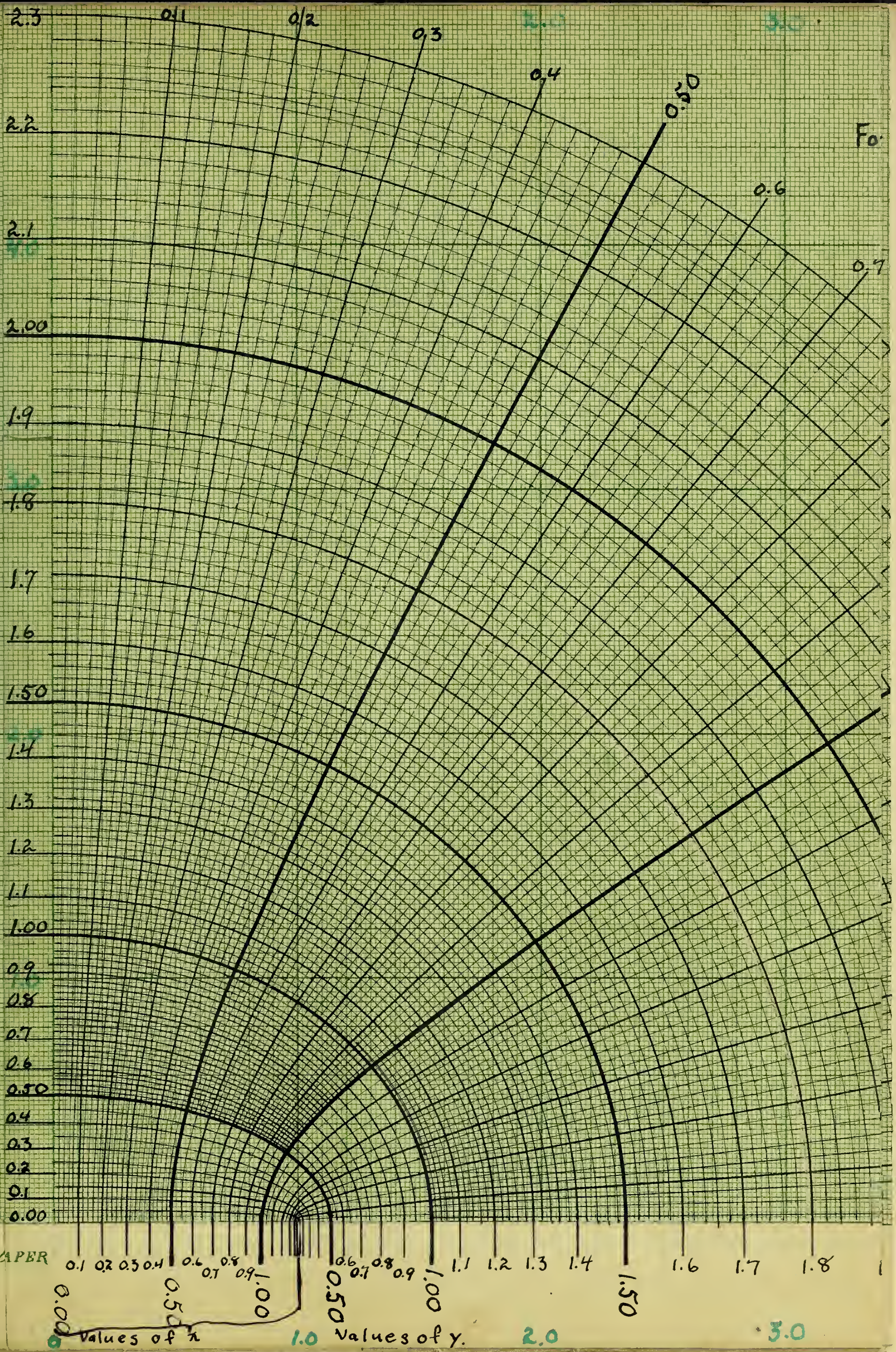
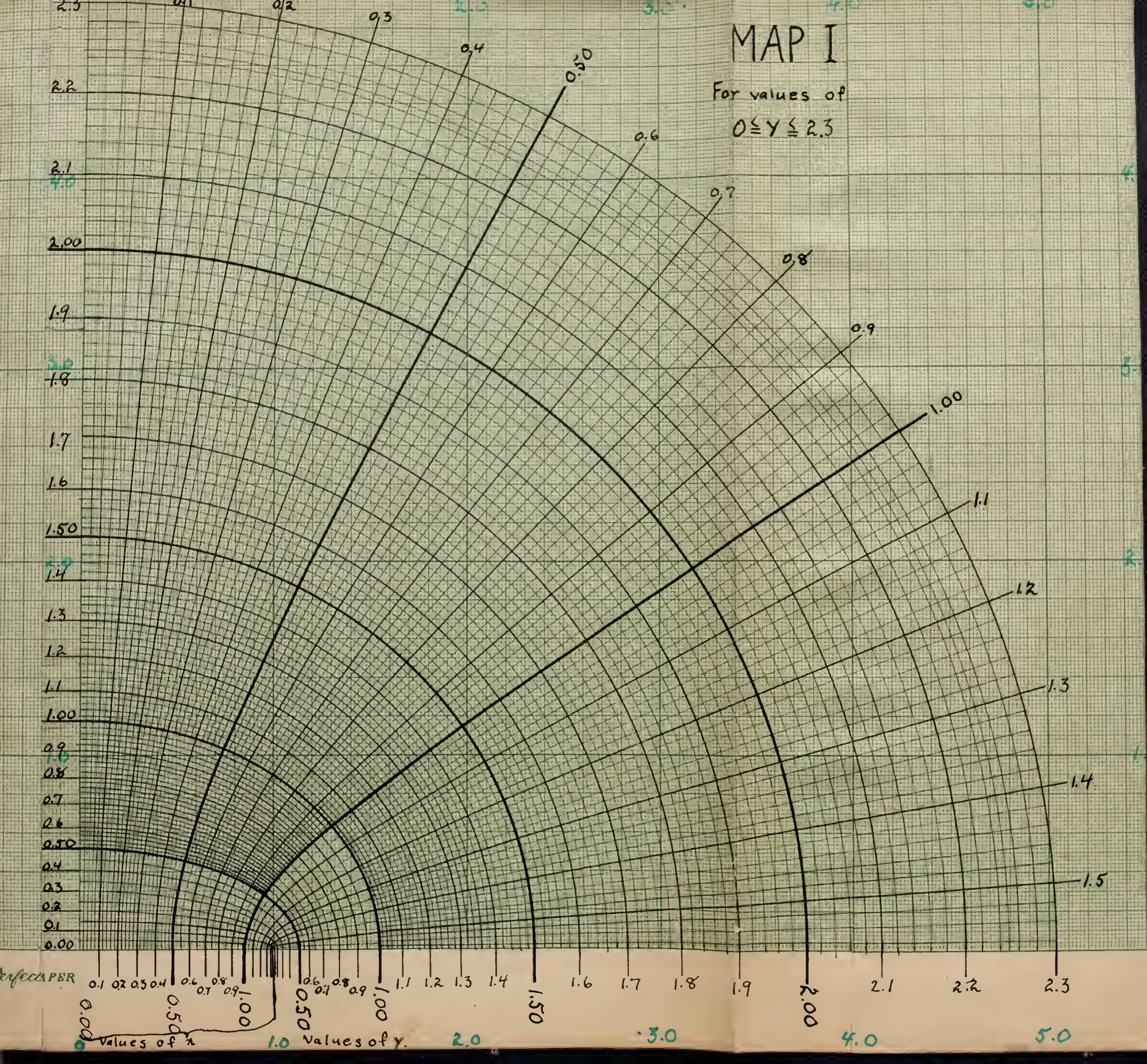
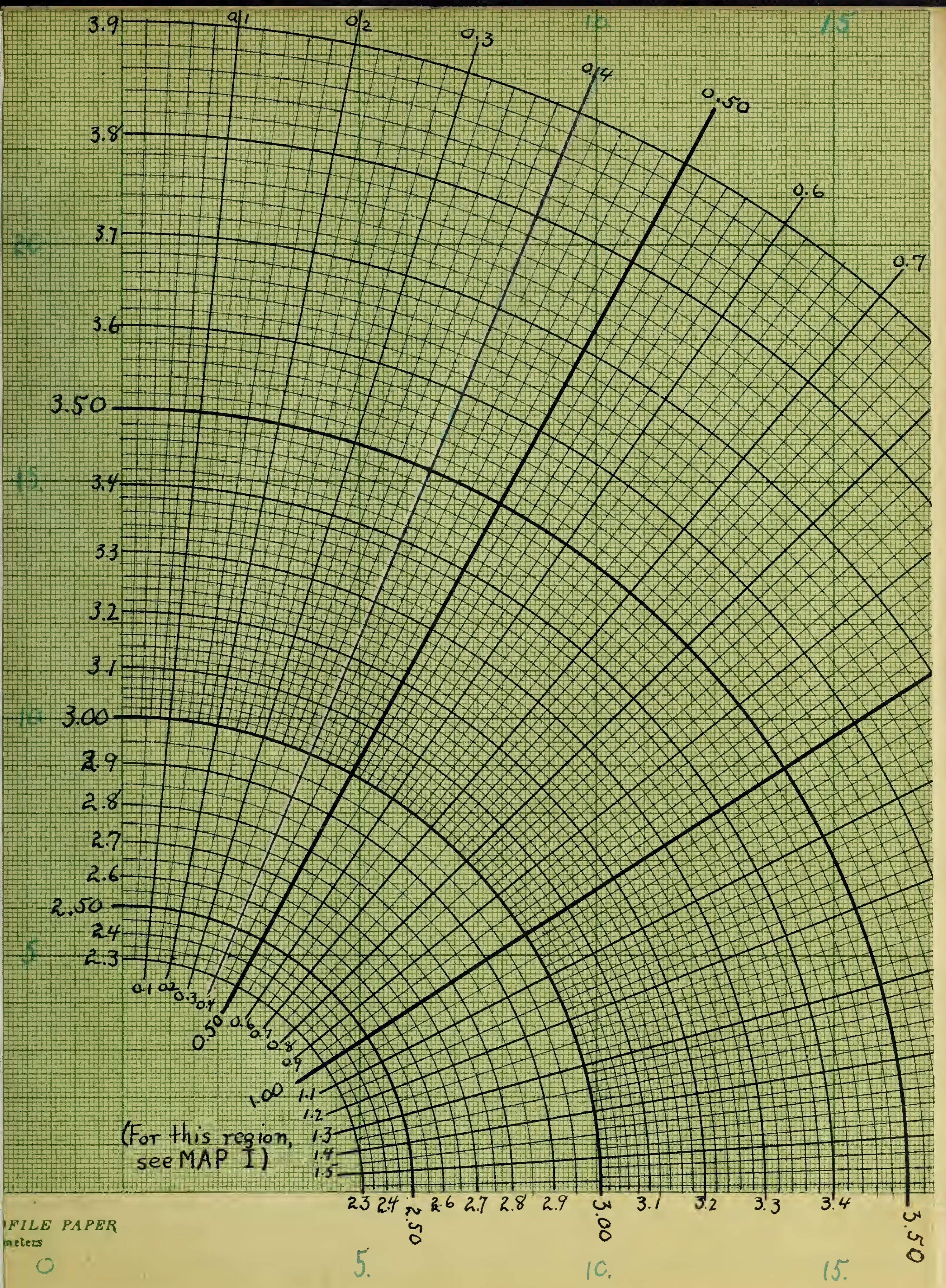
Blakesley, Thomas H., A table of hyperbolic cosines and sines, London, Taylor and Francis, 1890.

F I N I S

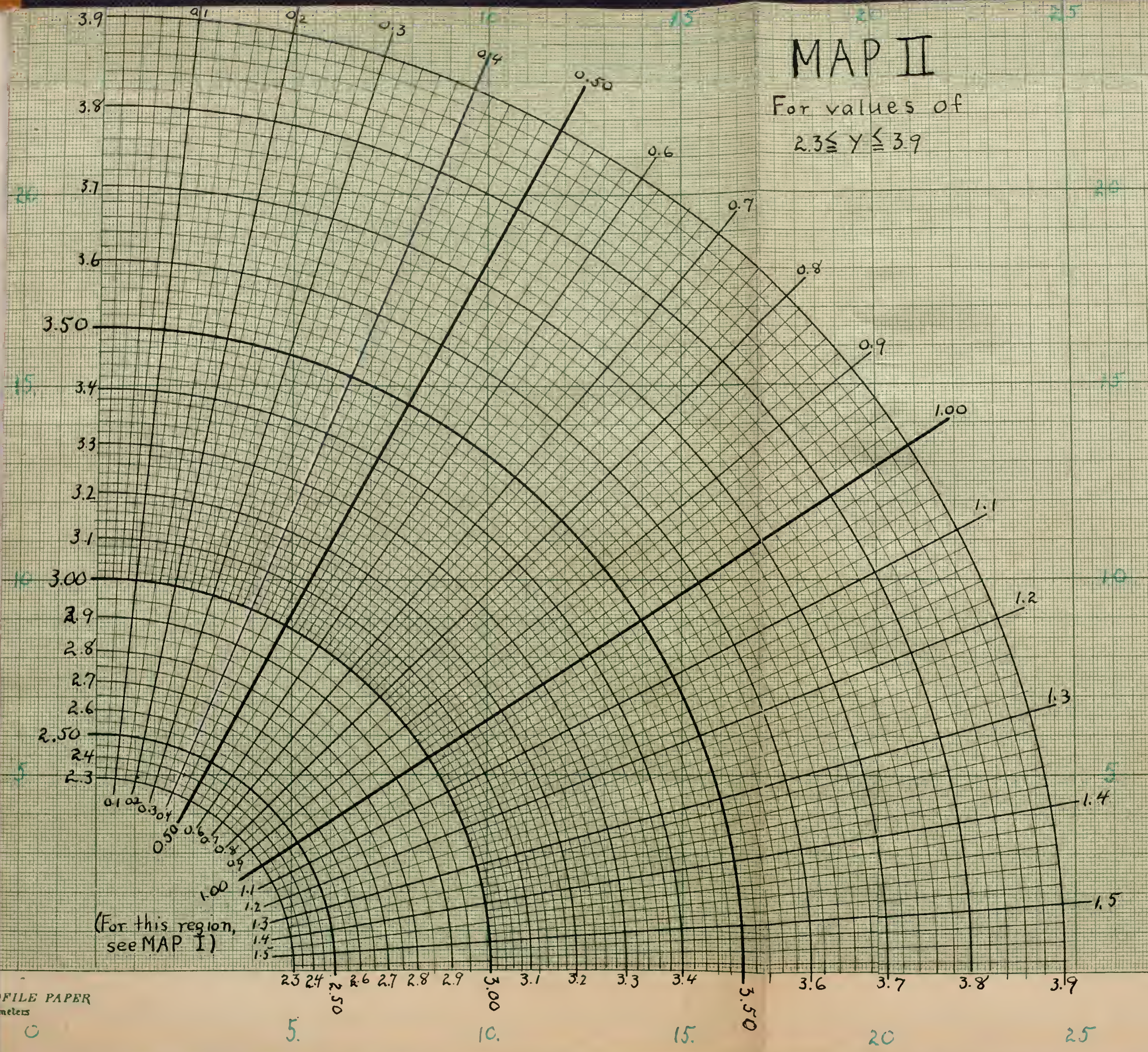


0.1
0.2
0.3
0.4
0.50
0.6
0.7
Fo
2.3
2.2
2.1
2.00
1.9
1.8
1.7
1.6
1.50
1.4
1.3
1.2
1.1
1.00
0.9
0.8
0.7
0.6
0.50
0.4
0.3
0.2
0.1
0.00
0.1 0.2 0.3 0.4 0.50 0.6 0.7 0.8 0.9 1.00
0.50 0.6 0.7 0.8 0.9 1.00 1.1 1.2 1.3 1.4 1.50 1.6 1.7 1.8
0.00
Values of x
Values of y.
0
1.0
2.0
3.0

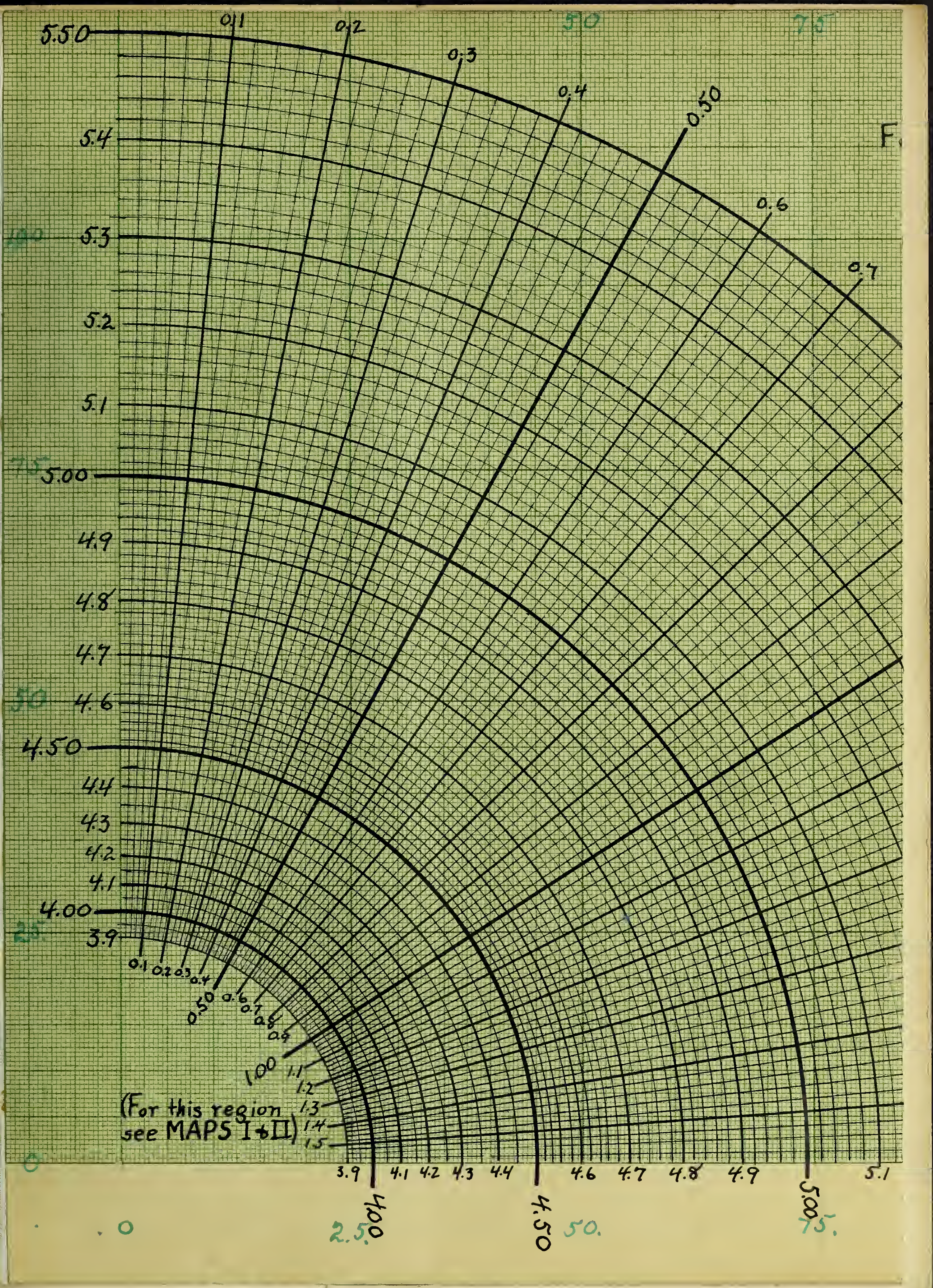

MAP I
For values of
0 ≦ y ≦ 2.3
0.1
0.2
0.3
0.4
0.50
0.6
0.7
0.8
0.9
1.00
1.1
1.2
1.3
1.4
1.5
Values of x
Values of y



3.9
3.8
3.7
3.6
3.50
3.4
3.3
3.2
3.1
3.00
2.9
2.8
2.7
2.6
2.50
2.4
2.3
0.1
0.2
0.3
0.4
0.50
0.6
0.7
0.8
0.9
1.00
1.1
1.2
1.3
1.4
1.5
(For this region, see MAP I)
2.3
2.4
2.50
2.6
2.7
2.8
2.9
3.00
3.1
3.2
3.3
3.4
3.50
FILE PAPER
meters
0
5.
10.
15.



MAP II
For values of
2.3 ≦ Y ≦ 3.9
(For this region, see MAP I)
0.1
0.2
0.3
0.4
0.50
0.6
0.7
0.8
0.9
1.00
1.1
1.2
1.3
1.4
1.5
2.3
2.4
2.50
2.6
2.7
2.8
2.9
3.00
3.1
3.2
3.3
3.4
3.50
3.6
3.7
3.8
3.9
FILE PAPER
meters



5.50
5.4
5.3
5.2
5.1
5.00
4.9
4.8
4.7
4.6
4.50
4.4
4.3
4.2
4.1
4.00
3.9
0.1
0.2
0.3
0.4
0.50
0.6
0.7
0.8
0.9
1.00
1.1
1.2
1.3
1.4
1.5
(For this region see MAPS I + II)
3.9
4.00
4.1
4.2
4.3
4.4
4.50
4.6
4.7
4.8
4.9
5.00
5.1
50
75
100
75
50
25
0
.0
2.5.
50.
75.
F.

# MAP III

For values of $3.9 \leqq \gamma \leqq 5.5$